Convergence Rates for Renewal Sequences

M. C. Spruill
School of Mathematics
Georgia Institute of Technology
Atlanta, Ga. USA

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ABSTRACT

The precise rate of geometric convergence of nonhomogeneous renewal sequences is characterized. Application is made to success runs, queueing theory, and simulated annealing.

Keywords: Geometric convergence, Queueing, Runs, Counters, Restarted simulated annealing

1. INTRODUCTION.

A renewal sequence u_n satisfies, for $n = 0, 1, \ldots,$ the discrete renewal equation,

$$u_n = b_n + \sum_{j=1}^{n} f_j u_{n-j}, \tag{1}$$

where $u_0 = b_0$. In the literature, the f's, b's and u's are typically scalars, often with conditions such as $f_j \geq 0$ for example, but our results shall apply under the more general assumption that the u's and b's are members of a complex Banach algebra A with unit e and the f's are either members of A or are complex scalars. In either case throughout this work we take, along with $u_0 = b_0$, $f_0 = 0$.

The scalar case is well studied. Conditions for convergence of the renewal sequence u_n and its limiting value have been known for a long time; see for example [2] or [4]. Numerous extensions have been studied. The discrete theory was extended to the continuous case in [1]. Karlin in [6] allowed n to be negative and considered convergence rates also. The paper [4] contains some results on rates. Kendall in [7] established conditions for geometric convergence of renewal sequences through his study of geometric ergodicity of countable Markov chains. For the relationship between renewal equations and Markov chains see [8]. Meyn and Tweedie make use in [10], for more general state spaces, of Kendall's results. Geometric convergence in equation (1) need not hold; see, for example, [11] or [12].

The precise rate of convergence when geometric convergence holds is of interest here and our result is the following.

Theorem 1. Let $a \in (0, 1)$. If

$$\liminf_{n\to\infty} ||f_n||^{-1/n} = r_f > 0 \tag{2}$$

and

$$\liminf_{n\to\infty} ||b_n||^{-1/n} = r_b > 0, \tag{3}$$

then there is an element $u \in A$ such that for each $t \in (a, 1)$,

$$\lim_{n\to\infty} t^{-n} ||u_n - u|| = 0 \tag{4}$$

if and only if

$$h(z) = [\frac{e - F(z)}{1 - z}]^{-1} B(z)$$

is holomorphic in $|z| < 1/a$. Furthermore, when (4) holds, $u = h(1)$.

Here and below the function of z denoted by a cap letter uses the small letters as $C(z) = \sum_{j\geq 0} c_j z^j$. The a of the theorem is the rate of geometric convergence.

In the next section the theorem is proved and in the one following is applied to some examples from probability. Included is characterization of the precise geometric rate of convergence of restarted simulated annealing (see[9]).

2. PROOF OF THEOREM 1

An element a of A is invertible if it has a two sided inverse. If g is a holomorphic A-valued function in a disc $|z| < \epsilon$ and the values of g in A are invertible then there is an inverse funtion g^{-1} which is also holomorphic in this disc (see [5]). It can happen that the radius of convergence of the latter exceeds that of the former and in that case the same symbol g^{-1} will be employed even though in the domain of definition of g^{-1} there are perhaps z's at which the very existence of $g(z)$ is in question. For example, in the scalar case the function $g(z) = \sum_{n\geq 0} z^n$ defined for $|z| < 1$ has inverse $g^{-1}(z) = 1 - z$ on $|z| < 1$, but the latter converges in the whole plane.

Lemma 2. If u_n satisfies (1), (2), and (3) then there is a $K \in (0, \infty)$ and a $C \in (0, \infty)$ such that $||u_n|| \leq CK^n$ for all n.

Proof: Let $0 < r < min\{r_f, r_b\}$. Then

$$F = \sum_{n\geq 1} r^n \|f_n\| < \infty$$

and setting

$$M_n = \max\{r^j \|u_j\| : 0 \leq j \leq n\},$$

one has

$$M_n \leq \max\{A^n + M_{n-1}F, M_{n-1}\},$$

where for some $A \in (0, \infty)$ and all n, $r^n \|b_n\| \leq A^n$. Assuming, without loss of generality, A and F greater than 1,

$$M_n \leq$$
$$A^n + F(A^{n-1} + FM_{n-2}) \leq$$
$$\cdots \leq V \sum_{j=0}^{n} A^{n-j} F^j \leq VnG^n,$$

where $G = \max\{F, A\}$. □

Lemma 3. Under the conditions (3) and (2) for some $\epsilon > 0$, the series U, F, and B converge for $|z| < \epsilon$ and the functions so represented there satisfy

$$U(z) = B(z) + F(z)U(z).$$

Proof: The convergence of the series follows from Lemma 2, (2) and (3) . The equation follows from (1) . □

Lemma 4. Under the conditions (3) and (2) one has for some $\eta > 1$

$$\sum_{n\geq 1} \|u_n - u_{n-1}\| \rho^n < \infty \tag{5}$$

for all $\rho \in (1, \eta)$ if and only if

$$[\frac{e - F(z)}{1 - z}]^{-1} B(z)$$

is holomorphic in $|z| < \eta$.

Proof: Assuming that $[\frac{e-F(z)}{1-z}]^{-1}B(z)$ is holomorphic in $|z| < \eta$, setting $u_n = 0$ for $n \leq -1$, and taking $y(n) = u_{n+1} - u_n$ one has, for $\epsilon > 0$ sufficiently small, by lemma 2, that $Y_1(z) = \sum_{n\geq 0} y(n-1)z^n = (1 - z)U(z)$ is holomorphic on $|z| < \epsilon$. Furthermore, by lemma 3, on a neighborhood of 0

$$B(z) = (e - F(z))U(z)$$
$$= \frac{e - F(z)}{1 - z}(1 - z)U(z)$$
$$= \frac{e - F(z)}{1 - z} Y_1(z).$$

It follows that $Y_1(z) = [\frac{e-F(z)}{1-z}]^{-1}B(z)$ is holomorphic in $|z| < \eta$ so, if $\rho < \eta$ then

$$\liminf_{n\to\infty} ||y(n-1)||^{-1/n} > \rho.$$

Taking $\rho < (1 + \delta)\rho < \eta$, for n sufficiently large

$$||u_n - u_{n-1}|| < [1/(1 + \delta/2)\rho]^n$$

and the series (5) converges.

Now suppose that the series (5) converges for all $\rho < \eta$, where $\eta > 1$ and let $\rho \in (1, \eta)$ be arbitrary. Since u_n is a Cauchy sequence in the Banach algebra it has a limit a. It also follows that $(U - a)(z) = \sum_{n\geq 0}(u_n - a)z^n$ is holomorphic in $|z| < \rho$. Now on a sufficiently small neighborhood of 0 one has $(U - a)(z) = U(z) - a/(1 - z)$ so that

$$B(z) = \frac{e - F(z)}{1 - z}(1 - z)U(z)$$
$$= \frac{e - F(z)}{1 - z}(1 - z)((U - a)(z)$$
$$+ a/(1 - z))$$

or

$$B(z) = \frac{e - F(z)}{1 - z}((1 - z)(U - a)(z) + a).$$

Since $F(0) = 0$, on a sufficiently small neighborhood of 0 ,$\frac{e-F(z)}{1-z}$ has a holomorphic inverse. Therefore

$$[\frac{e - F(z)}{1 - z}]^{-1}B(z)$$

$$= (1 - z)(U - a)(z) + a$$

on that small disk and it follows that

$$[\frac{e - F(z)}{1 - z}]^{-1}B(z)$$

is holomorphic in $|z| < \rho$ since the function on the right hand side is holomorphic there. □

Proof of Theorem 1: If h is holomorphic in $|z| < 1/a$ then by lemma 4, u_n is a Cauchy sequence in the Banach algebra with a limit u and for $\rho < 1/a$

$$||u_{n+m} - u_n|| \leq \sum_{j=0}^{m-1} ||u_{n+j+1} - u_{n+j}||$$

$$\leq K\rho^{-(n+1)}/(1 - \rho).$$

Taking limits on $m \to \infty$ above shows that for all n, $||u_n - u|| \leq M\tau^n$, where $\tau = 1/\rho > a$. Furthermore, since $Y_1(z) = [\frac{e-F(z)}{1-z}]^{-1}B(z) = h(z)$ is holomorphic in a region containing 1, one has

$$\lim_{z\to 1} Y_1(z) = h(1)$$

and, from the definition of $Y_1(z)$,

$$\lim_{z\to 1} Y(z) = \lim_{n\to\infty} u_n.$$

Conversely, suppose that (4) holds. Then lemma 4 shows that

$$[\frac{e - F(z)}{1 - z}]^{-1}B(z) = h(z)$$

is holomorphic in $|z| < 1/a$ and the argument just given concerning the value of the limit applies. □

3. Examples

Example 1. Geiger Counters (see [3]): Bernoulli trials are performed with success probability p. A counter meant to register successes is locked for exactly $r - 1$ epochs following a registration so the occurrence of a success at epoch n would be registered if no registration of one occurred in the preceding $r - 1$ epochs. Upon a registration of a success at epoch n the mechanism would then be locked for epochs $n + 1, \dots, n + r - 1$, so that a success could not be registered during any of these. A success would be registered at the $n + r$ th epoch if it occurred since the counter would then be free. Let w_n be the probability the counter is free (will register a success if one occurs) at epoch $n, n \geq 1$, where $w_1 = 1$. If the probability of success p lies in $(0, 1)$ then there is, for each ρ greater than the reciprocal of the smallest magnitude of those roots of $1-(qz+pz^r) = 0$ outside the unit circle, a finite K such that for all n

$$|w_n - 1/(q + rp)| \leq K\rho^n,$$

while for for every ρ less than that reciprocal there is no choice of real K for which this holds.

To see this, begin by setting $f_j = q$ for $j = 1$, $f_j = p$ for $j = r(\geq 2)$, $f_j = 0$ otherwise, and observing that for all $n \geq 2$

$$w_n = \sum_{j=1}^{n} f_j w_{n-j}.$$

Taking $u_n = w_{n+1}, b_0 = u_0 = 1$, and $b_j = 0 = u_{-j}$ for $j \geq 1$, one has for $n \geq 1$

$$u_n = \sum_{j=1}^{n+1} f_j u_{n-j} + b_n$$

$$= \sum_{j=1}^{n} f_j u_{n-j} + b_n$$

so that

$$F(z) = \sum_{n\geq 1} f_n z^n = qz + pz^r$$

and

$$[\frac{e - F(z)}{1 - z}]^{-1}B(z)$$

$$= \frac{1 - z}{1 - (qz + pz^r)}$$

for $|z|$ sufficiently small. For $p \in [0, 1]$ the function $1 - (qz + pz^r)$ always has all of its r roots on or outside of the unit circle. If the single root $z = 1$ is on the unit circle then the rate of convergence of w_n to $h(1) = -1/(-q - (r)p) = 1/(q + rp)$ is $1/\eta$, where η is the smallest magnitude of the roots outside the circle. Since for any $e^{i\theta}$, $pe^{ir\theta} + qe^{i\theta}$ is a convex combination of extreme points of the (convex) unit disk, an interior point unless the points coincide, whenever $p \in (0, 1)$, $1 - (qz + pz^r)$ always has its only roots on the unit circle at $z = 1$. □

Example 2. Success runs (see [3]) in Bernoulli trials. In a sequence of S's and F's the number of runs of length r in n trials is the number of non-overlapping blocks containing a sequence of exactly r uninterrupted S's each. For example, the sequence $SSFSSSFFSSFFSSS$ has two runs of length three and four of length 2. For r fixed, a regenerative event E occurs at the n^{th} epoch if the outcome at that trial yields a new run of length r. Letting $u_n = P[$ E on epoch n], it is shown that, for any $\rho > p$ there is a $K < \infty$ such that for all n

$$|u_n - \frac{p^r q}{1-p^r}| \leq K\rho^n$$

while this fails for any $\rho < p$. To verify this, observe that for $n \geq r$,

$$u_n + pu_{n-1} + \cdots + p^{r-1}u_{n-r+1} = p^r,$$

while

$$u_0 = 1, u_1 = \cdots = u_{r-1} = 0.$$

Thus

$$u_n = b_n + \sum_{j=1}^{n} f_j u_{n-j}$$

and $u_0 = b_0$, where $b_n = p^n$ for $0 \leq n \leq r$, $b_n = p^r$ for $n \geq r+1$, and $f_n = -p^n$, for $1 \leq n \leq r-1$, and $f_n = 0$ otherwise. One has for $|z|$ sufficiently small

$$[\frac{1-F(z)}{1-z}]^{-1}B(z)$$

$$= \frac{1-z+p^r q z^{r+1}}{1-(pz)^r}$$

$$= -qz + \frac{pz-1}{(pz)^r - 1}.$$

This rational function has all of its $r-1$ poles on the circle of radius $1/p$ and

$$h(1) = \frac{p^r q}{1-p^r}. \qquad \square$$

Example 3. The GI/M/1 queue: In the GI/M/1 queuing model, arrivals are independent according to a general distribution, service is according to an (independent) exponential distribution and there is a single server who is idle only when the queue is empty. There will be an asymptotic distribution of numbers of customers in the queue when it starts empty. The rate of convergence to this distribution is determined here. Denoting by X_n the number of customers in the queue, waiting or being served, at the n^{th} arrival, and with τ the mean service time, one has for $0 < y \leq x+1$,

$$p_{x,y} = P[X_{n+1} = y | X_n = x] = g_{x+1-y},$$

$$p_{x,0} = \sum_{j \geq x+1} g_j,$$

where

$$g_n = \int_0^\infty e^{-s/\rho} \frac{(s/\rho)^n}{n!} dA(s),$$

$\rho = \tau/\mu =$ mean service time/mean arrival time, and A is the arrival distribution. In the subcritical case $\rho < 1$ geometric convergence takes place to a non-degenerate distribution (see [7]). Theorem 1.1 can be used to establish the precise rate at which the probability of an empty queue approaches its stationary value in the subcritical case when the queue starts empty as follows.

Let $W_1, W_2, \ldots$ be iid random variables with $P[W_j = i\ ] = g_i$ and $S_m = \sum_{j=1}^m W_j$. Also, let, for $i \geq 1$, f_i be the probability of first hitting state 0 at epoch i given a start in state 0 and set $r_j = \sum_{i \geq j+1} f_i$. The probability u_n of being in state 0 at epoch n satisfies, for $n \geq 0$, the renewal equation

$$u_n = b_n + \sum_{j=1}^{n} f_j u_{n-j}$$

with $u_0 = b_0 = 1$, $b_n = 0$ for $n \geq 1$. In the subcritical case $\rho < 1$ the rate at which the probability of an empty queue approaches its stationary value when the queue starts empty is

$$\limsup_{m\to\infty} P^{1/m}[S_m = m-1]. \tag{6}$$

This is a consequence of the fact that with $R(z) = \sum_{n\geq 0} r_n z^n$ and when $\rho < 1$,

$$1 - \sum_{n\geq 1} f_n z^n = (1-z)R(z),$$

so that the radius of convergence of $[\sum_{j\geq 0} r_j z^j]^{-1}$ is the required quantity. Kendall shows that

$$[\sum_{j\geq 0} r_j z^j]^{-1} = 1 - \sum_{n\geq 1} t_n z^n,$$

where

$$t_n = \frac{1}{2\pi i n}\oint_C \frac{G^n(z)}{z^n} dz,$$

and C is any suitable closed curve about zero in the positive sense. Here

$$G(z) = \sum_{n\geq 0} g_n z^n.$$

Noting that nt_n is simply the coefficient of z^{n-1} in the expansion of G^n and that

$$G^n(z) = E[z^{S_n}],$$

one has $nt_n = P[S_n = n-1]$ confirming (6) . This limit can be evaluated explicitly and a simple expression of the limit found for the subcritical GI/M/1 queue with arrivals according to the gamma (or Erlang) distribution, $G(k,b)$ whose density is

$$f(t) = \frac{t^{k-1}e^{-t/b}}{\Gamma(k)b^k}.$$

In this case, the rate can be seen to be

$$\rho^k [\frac{k+1}{1+k\rho}]^{k+1},$$

where $\rho = \tau/kb$ = mean service time/mean interarrival time by applying Stirling's approximation of $n!$ to $P[S_n = n-1]$. This is true, since in this case we have

$$\phi_W(z) = [\frac{k\rho}{k\rho + 1 - z}]^k,$$

the probability generating function of a negative binomial with kn successes, so that

$$P[S_n = n-1] = \binom{nk+n-1}{nk-1} p^{nk} q^{n-1}.$$

Example 4. Simulated annealing for finding the minimum of a function f on a set C generates a C-valued stochastic process X_n whose (non-stationary) law of motion is given by

$$P[X_{n+1} = y \ |X_n = x \ ]$$
$$= q_{xy} \ \min\{1, e^{-(f(y)-f(x))/T(n)}\},$$

where $x \neq y$ are in C, q_{xy} are the transition probabilities of a fixed C-valued Markov chain, and $T(n)$ is a function strictly decreasing to 0. Mendivil, Shonkwiler, and Spruill [9] restarted the simulated annealing whenever the vector process

$$\Pi_n = (X_n, X_{n+1}, \ldots, X_{n+r})$$

lay in a subset D of the product space C^r. The subset they studied was $D = \{(x_1, \ldots, x_r) \in C^r : f(x_1) = \cdots = f(x_r)\}$. They showed that with $E \subset C$ the set of basin bottoms (see the paper for precise definitions; it is the maximal collection of local minima under a neighborhood system dictated by the Markov transition q_{xy} and the function f) and τ_A for $A \subset C$ the first hitting time by the process Π_n of the set

$$D_A = \{v \in C^r : v_1 \in A\},$$

the first hitting time of the goal, τ_G, has tail probability $u_n = P[\tau_G > n]$ satisfying a renewal equation

$$u_n = b_n + \sum_{j=1}^{n} P[\{\tau_G > n\} \cap \{\tau_U = j\}]$$
$$= b_n + \sum_{j=1}^{n} u_{n-j} f_j,$$

where $U = G \cup E$,

$$f_n = P[\tau_G > n, \tau_U = n],$$

and $b_n = P[\tau_U > n]$. They showed, furthermore and in the notation here, that the radius of convergence of $B(z)$ (and hence $F(z)$) is greater than 1, and that under their restarting scheme $F(1) < 1$. It follows from the result here that $h(1) = 0$ and the rate of convergence of the tail probabilities $P[\tau_G > n]$ to 0 is the reciprocal of the radius of convergence of

$$\frac{B(z)(1-z)}{1-F(z)}.$$

Consulting Corollary 3.2 of [9], one finds an expression there for the precise rate under some conditions. They appealed there to the result Theorem XIII.10.1 of [3] which requires a real solution $\theta > 1$ to $F(\theta) = 1$ and $B(\theta) < \infty$. The expression here is more general; neither of the two requirements of Feller's theorem are necessary. Compare with Example 2 above in which our theorem yields a rate but the Feller theorem does not. Feller establishes a rate for the problem of Example 2 using generating function arguments whereas our Theorem 1 yields the rate immediately. □

References

[1] D. Blackwell, A renewal theorem. Duke Math. J. 15, 1948, pp. 145–150.

[2] P. Erdos, W. Feller, and Pollard, H. A property of power series with positive coefficients. Bull. Amer. Math. Soc., 55, 1949, pp. 201-204.

[3] W. Feller, An Introduction to Probability and Its Applications, pg. 332 of Vol I, 1968.

[4] W. Feller, Fluctuation Theory of Recurrent Events. Trans. Amer. Math. Soc. 67, 1949, pp. 98-119.

[5] R. V. Kadison, and J. V. Ringrose, Fundamentals of the Theory of Operator Algebras, Academic Press, New York,1983 .

[6] S. Karlin, On the renewal equation, Pacific Jnl. of Math., 5, 1955, pp. 229-257.

[7] D. G. Kendall, Geometric ergodicity and the theory of queues. In Mathematical Methods in the Social Sciences, Arrow, Karlin, and Suppes, eds., Stanford, 1960.

[8] J. F. C. Kingman, Regenerative Phenomena, Wiley, 1972.

[9] F. Mendivil, R. Shonkwiler, and M. C. Spruill, Restarting search algorithms with applications to simulated annealing, Adv. Appl. Prob. 33, 2001, pp. 242-259.

[10] S. P. Meyn and R. L. Tweedie, Markov Chains and Stochastic Stability. Springer-Verlag, London, 1993.

[11] J. W. Pitman, Uniform rates of convergence for Markov chain transition probabilities, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 29, 1974, pp. 193-227.

[12] C. Stone, On characteristic functions and renewal theory, Trans. Amer. Math. Soc. 120,1965, pp. 327-342.